THE POWERSET ALGEBRA AS A RESULT OF ADDING PROGRAMMING CONSTRUCTS TO THE NESTED RELATIONAL ALGEBRA

Marc Gyssens, University of Antwerp (UIA), B-2610 Antwerpen, Belgium
Dirk Van Gucht, Indiana University, Bloomington, IN 47405-4101, USA

In this paper, we discuss augmentations of the nested relational algebra with programming constructs, such as while-loops and for-loops. We show that the algebras obtained in this way are equivalent to a slight extension of the powerset algebra, thus emphasizing both the strength and the naturalness of the powerset algebra as a tool to manipulate nested relations, and, at the same time, indicating more direct ways to implement this algebra.

1. Introduction

In the last years, much attention has been paid to structured relations. In order to model some database applications more naturally, Makinouchi proposed to generalize the relational model by removing Codd's first normal form assumption [7], thus allowing relations with set-valued attributes [16]. Subsequently, a generalization of the relational algebra to relations with set-valued attributes was introduced by Jaeschke and Schek [14]. More specifically, they presented the nest and the unnest operator as tools to restructure such relations. Finally, Thomas and Fischer generalized this model by allowing nested relations of arbitrary depth [21]. Calculus like query languages for such models were defined, e.g. in [1,17,19].

In an algebraic language used in [15], Kuper and Vardi introduced the powerset operator. Recently, much attention has been paid to the powerset algebra obtained by adding this operator to the nested algebra (e.g. [1,10,13]). This algebra was found to be considerably more expressive than the nested algebra [10,12]. Another extension of the nested relational algebra that has been considered, is its least fixpoint closure [2,10], which is equivalent to the powerset algebra.

In [4], Chandra and Harel introduced a very powerful query language for classical relations, called QL. In [5], they discussed RQL, which is a restriction of QL. Both languages contain programming constructs, such as while-loops. They showed that RQL is at least as expressive as the least fixpoint closure of relational query languages. They furthermore observed that showing that RQL is equivalent in expressive power to the fixpoint closure of a relational query language would imply that $PTIME = PSPACE$, which is generally believed to be false.

In this paper, we add programming constructs such as while-loops and for-loops to the nested algebra and show that these extensions yield a query language equivalent to the powerset algebra and hence also the least fixpoint closure of the nested relational algebra, thus underlining the strength and the naturalness of this algebra, establishing a sharp distinction between the properties of query languages for the nested relational model and the standard relational model, and, finally, indicating ways to implement the powerset algebra.

2. A model for working with nested relations

2.1. Nested relations

Basically we assume that we have an infinitely enumerable set U of *elementary attributes* and an infinitely enumerable set V of *elementary values*. In this section, we explain how arbitrary attributes and values, relation schemes, relation instances and relations are constructed from these.

First, we define an attribute. Attributes can either be elementary or composed. The latter ones are sets of attributes (which can be composed in turn); the values associated to them are relation instances over that set of attributes, interpreted as a scheme.

Definition 2.1.1

The set of all attributes $\mathcal{U}$ is the smallest set containing U such that for each finite subset X of $\mathcal{U}$ in which no elementary attribute appears more than once, $X \in \mathcal{U}$.

An attribute of U is called an *elementary attribute*; an attribute of $\mathcal{U} - U$ is called a *composed attribute*.

The values associated to composed attributes will be called quite fittingly composed values. The definition of a *relation scheme* is now very straightforward:

Definition 2.1.2

A relation scheme Ω is a composed attribute, i.e. an element of $\mathcal{U} - U$.

Example 2.1.1

Consider a relation representing persons, their jobs and the cities in which these jobs are executed.

PERSON	{{*JOB*	} *CITY* }
Jeff Willows	professor president	Austin
	consultant	Dallas
Mary Higgins		

There are two levels of nesting: jobs are grouped by the city in which they are executed and pairs of sets of jobs and cities are grouped by the person having these jobs. The scheme of the above relation is a set of two attributes, the former of which is the elementary attribute *PERSON* whereas the latter is the composed attribute $\{\{JOB\}, CITY\}$.

Let us now explain on this example, how we intend to define relation instances. The instance of the above relation will consist of two tuples. In each of these tuples, the value corresponding to *PERSON* is atomic whereas the composed value corresponding to $\{\{JOB\}, CITY\}$ is in turn a relation instance over the scheme $\{\{JOB\}, CITY\}$. In the former tuple, this relation instance again consists of two tuples. In each of these tuples, the value corresponding to $\{JOBS\}$ is a relation instance over $\{JOBS\}$ whereas the value corresponding to *CITY* is atomic. In the latter tuple the aforementioned relation instance is empty.

It should be clear by now that the notions of value, tuple and instance are so closely intertwined that it is easier to define them jointly:

Definition 2.1.3

The set $\mathcal{V}$ of all values, the set $\mathcal{I}_X$ of all instances over $X \in \mathcal{U} - U$, the set $\mathcal{T}_X$ of all tuples over $X \in \mathcal{U} - U$ and the set $\mathcal{I}$ of all instances are the smallest sets satisfying:

1. $\mathcal{V} = V \cup \mathcal{I}$;
2. $\mathcal{I} = \bigcup_{X \in \mathcal{U}-U} \mathcal{I}_X$;
3. *$\mathcal{I}_X$ consists of all finite subsets of $\mathcal{T}_X$;*
4. *$\mathcal{T}_X$ consists of all mappings t from X into $\mathcal{V}$, called tuples, satisfying $t(A) \in V$ for all elementary attributes $A \in X \cap U$ and $t(Y) \in \mathcal{I}_Y$ for all composed attributes $Y \in X - U$.*

We now have all the necessary ingredients to formally define a relation:

Definition 2.1.4

A relation is a pair (Ω, ω) where $\Omega \in \mathcal{U} - U$ and $\omega \in \mathcal{I}_\Omega$. Ω is called the scheme of the relation and ω is called the instance of the relation. If $\Omega \subseteq U$, then (Ω, ω) is called a flat relation.

Now let $\mathcal{W}$ be a set of relation instances over Ω. For notational convenience, we shall denote by $(\{\Omega\}, \mathcal{W})$ the relation with scheme $\{\Omega\}$ and instance $\{t \in \mathcal{T}_{\{\Omega\}} \mid t(\Omega) \in \mathcal{W}\}$. Note that there is no essential difference between a relation instance over a singleton scheme consisting of one composed attribute, and a set of relation instances over that composed attribute regarded as a scheme. We shall make thankful use of this dualism in the sequel.

2.2. The nested relational algebra

In this section, we define a nested algebra based on the model for relations described in the previous section. It is generated by eight operators. Basically, these operators are borrowed from the classical "flat" relational algebra, except for the nesting and unnesting. However, some technicalities were unavoidable to fit them in into our formalism.

Definition 2.2.1

Let (Ω, ω), (Ω, ω_1), (Ω, ω_2), (Ω_1, ω_1) and (Ω_2, ω_2) be relations. Suppose that the sets of elementary attributes from which Ω_1 and Ω_2 are built, are disjoint.

- *The union $(\Omega, \omega_1) \cup (\Omega, \omega_2)$ equals $(\Omega, \omega_1 \cup \omega_2)$;*
- *The difference $(\Omega, \omega_1) - (\Omega, \omega_2)$ equals $(\Omega, \omega_1 - \omega_2)$;*
- *The cartesian product $(\Omega_1, \omega_1) \times (\Omega_2, \omega_2)$ equals (Ω', ω') where $\Omega' = \Omega_1 \cup \Omega_2$ and*

$$\omega' = \{t \in \mathcal{T}_{\Omega'} \mid t|_{\Omega_1} \in \omega_1 \ \&\ t|_{\Omega_2} \in \omega_2\}$$

- *Let $\Omega' \subseteq \Omega$. The projection $\pi_{\Omega'}(\Omega, \omega)$ equals (Ω', ω') where $\omega' = \{t|_{\Omega'} \mid t \in \omega\}$;*
- *Let $X \subseteq \Omega$. The nesting $\nu_X(\Omega, \omega)$ equals (Ω', ω') where $\Omega' = (\Omega - X) \cup \{X\}$ and*

$$\omega' = \{t \in \mathcal{T}_{\Omega'} \mid \exists t' \in \omega : t|_{\Omega - X} = t'|_{\Omega - X} \ \&\ t(X) = \{t''|_X \mid t'' \in \omega \ \&\ t'|_{\Omega - X} = t''|_{\Omega - X}\}\}$$

- *Let $X \in \Omega - U$. The unnesting $\mu_X(\Omega, \omega)$ equals (Ω', ω') where $\Omega' = (\Omega - \{X\}) \cup X$ and*

$$\omega = \{t \in \mathcal{T}_{\Omega'} \mid \exists t' \in \omega : t|_{\Omega - \{X\}} = t'|_{\Omega - \{X\}} \ \&\ t|_X \in t'(X)\}$$

Let (Ω, ω) be a relation scheme. Let φ be a permutation on U. φ is extended in the natural way to $\mathcal{U}$, to $\mathcal{I}$ and to $\mathcal{V}$:

- *The renaming $\rho^\varphi(\Omega, \omega)$ equals $(\varphi(\Omega), \varphi(\omega))$;*

- *Assume furthermore that* $\varphi(\Omega) = \Omega$. *The selection* $\sigma^{\varphi}(\Omega,\omega)$ *equals* (Ω,ω') *where*

$$\omega' = \{t \in \omega \mid \forall X \in \Omega: \varphi(t(X)) = t(\varphi(X))\}$$ ∎

Example 2.2.1

Reconsider the relation in Example 2.1.1. If we denote this relation by (Ω,ω), then $\mu_{\{\{JOB\},CITY\}}(\Omega,\omega)$ yields a relation with scheme $\Omega' = \{PERSON, \{JOB\}, CITY\}$ which can be represented as:

PERSON	{*JOB* }	*CITY*
Jeff Willows	professor president	Austin
Jeff Willows	consultant	Dallas

If we nest this last relation over $\{\{JOBS\}, CITY\}$, we again obtain a relation over Ω the instance of which corresponds to the first tuple of ω. Note that in general, an unnesting cannot be undone by a nesting, even if no empty composed values are present. A nesting on the other hand, can always be undone by the corresponding unnesting. ∎

Note that the cartesian product is only defined for relations with completely "independent" schemes. This is actually not a severe restriction: it is indeed always possible to arrange that the schemes of two relations have no elementary attributes in common by performing an appropriate renaming.

We end this discussion about the basic nested algebra operators with a notational issue. In most practical cases, renaming involves only one attribute X at the time. If X is renamed to X', and if, in case X and X' are composed attributes, no ambiguity is possible as to how the renaming is done, we shall denote this operation as $\rho_{X' \leftarrow X}$. We shall use the same notation if X is a set of attributes of the scheme under consideration, and each attribute of X is renamed in a well known way to an attribute of X'. Similarly, if selection comes down to only checking whether the values for composed attributes X and X' are equal upon renaming and if no ambiguity is possible as to how the elementary attributes in X and X' are to be matched, we shall denote this selection by $\sigma_{X=X'}$. Again, we shall use the same notation if X and X' are sets of attributes of the scheme under consideration.

We can now define a *nested algebra expression (nae)*:

Definition 2.2.2

1. *The variables* $x, y, z, \ldots$ *are naes;*
2. *For all* $\Omega \in \mathcal{U} - U$, $(\Omega, \emptyset)$ *is a nae;*
3. *For all* $\Omega \in \mathcal{U} - U$, $(\{\Omega\}, \{\emptyset\})$ *is a nae;*
4. *For all naes, the basic operators of Definition 2.2.1 applied to them, are also naes, provided these new expressions make sense.* ∎

We implicitly assume that all variables are typed, i.e. associated to relations having one particular scheme. However, we shall not explicitly use this typedness for convenience of notation. The expressions introduced in items 1, 2 and 3 of Definition 2.2.2 will be called *primitive expressions*.

The set of all naes will be denoted by $\mathcal{N}$. If $E(x,y,\ldots) \in \mathcal{N}$ and $r,s,\ldots$ are a finite sequence of relations the schemes of which are "compatible" with the variables of the expression, then $E(r,s,\ldots)$ is interpreted as the relation obtained by substituting every occurrence of a variable in $E(x,y,\ldots)$ by the corresponding relation. Obviously, the scheme of $E(r,s,\ldots)$ does not depend on the actual instances of $r,s,\ldots$. Therefore we shall often denote this scheme as Ω^E.

To avoid extensive use of brackets, we assume the following precedence on nested algebra operators: unary operators, cartesian product, set operators.

3. The powerset algebra

Recently, much attention has been paid to the expressiveness of the nested relational algebra [1,13,18]. In order to deal with this problem, it suffices to consider single relations only, since a database can always be represented as the cartesian product of its non-empty members. In its most general form, the question that must be asked, is [4,5]: *Let* Q *be a computable query, i.e. a partial recursive mapping from relations to relations that preserves isomorphism. Does there exist* $E(x) \in \mathcal{N}$ *such that* $\forall r: E(r) = Q(r)$? Although it has been shown [8,22] that it is always possible to find an expression that satisfies this equality for any particular relation, there is in general no expression that will do for all relations. E.g. the transitive closure of a binary flat relation, which is not expressible in the classical relational algebra, is also not expressible in the nested relational algebra [18].

Therefore, several attempts have been made to enrich the nested relational algebra. One of these consists of adding the powerset operator to the nested algebra. This operator was introduced by Kuper and Vardi in [15] as one of the primitive operators in their algebraic query language for database logic. Basically the powerset operator generates all subsets of a given relation:

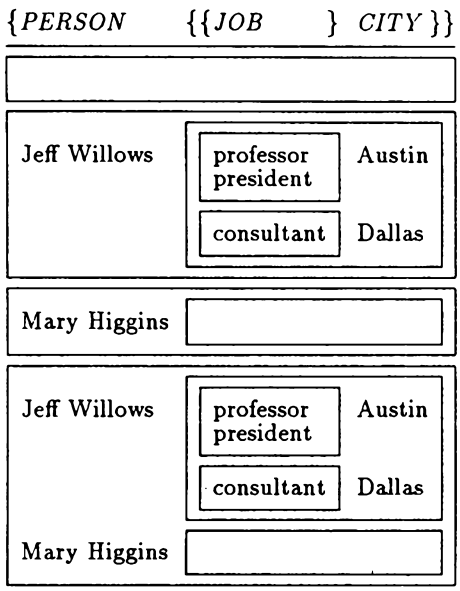
Definition 3.1

Let (Ω,ω) *be a relation. Let* 2^{ω} *denote the set of all subsets of* ω. *Then (using the notation introduced earlier) the powerset* $\Pi(\Omega,\omega)$ *equals* $(\{\Omega\}, 2^{\omega})$. ∎

Example 3.1

Reconsider the relation (Ω,ω) represented in Example 2.1.1. $\Pi(\Omega,\omega)$ is a relation with scheme $\{\{PERSON,$

$\{\{JOB\}, CITY\}\}\}$ which can be represented as:



Using a combinatorial argument, it was established in [10] (a similar result was obtained independently by Houben, Paredaens and Tahon [12]):

Theorem 3.1

There is no nae that expresses the powerset operator.

We now consider the *powerset algebra* whose set of expressions $\mathcal{P}$ is generated by the basic operators of the nested relational algebra defined in Definition 2.2.1, augmented with the powerset operator. An expression of the powerset algebra is called a *powerset algebra expression (pae)*. The expressiveness of the powerset algebra is illustrated in part by a result of [10]:

Theorem 3.2

In the powerset algebra, either the nest operator or the difference can be considered as redundant.

4. The least fixpoint operator

The *least fixpoint operator* [3,5,6] actually does not work on relations but on queries; it transforms them into other ones. We first point out for which queries we shall consider the least fixpoint operator:

Definition 4.1

An lfp expression is a unary scheme preserving expression $E(x)$ such that for all relations r and s for which $E(r)$ and $E(s)$ are defined:

1. *$r \subseteq E(r)$ (increasing);*
2. *$r \subseteq s$ implies $E(r) \subseteq E(s)$ (monotone).*

If $E(x)$ is an lfp expression and r is a relation, then $E^*(r)$ is defined if and only if $E(r)$ is defined and must in that case be interpreted [20] as the smallest relation s containing r for which $E(s) = s$. A straightforward argument shows that $E^*(r) = \bigcup_{i=1}^{\infty} E^i(r)$. Note that for each relation r for which $E(r)$ is defined, $E^*(r)$ can always be computed, since for some positive integer k, $E^k(r) = E^{k+1}(r) = E^{k+2}(r) = \cdots$.

We now formally define the lfp closure of the nested relational algebra (respectively the powerset algebra):

Definition 4.2

The lfp closure of the nested algebra $\mathcal{N}$ (respectively the powerset algebra $\mathcal{P}$) is the smallest set $\mathcal{N}^*$ (respectively $\mathcal{P}^*$) satisfying:

1. *$\mathcal{N}^*$ (respectively $\mathcal{P}^*$) contains all primitive expressions;*
2. *$\mathcal{N}^*$ (respectively $\mathcal{P}^*$) is closed under the basic operators of the nested algebra (respectively the powerset algebra);*
3. *For each lfp expression $E(x)$ in $\mathcal{N}^*$ (respectively $\mathcal{P}^*$), $E^*(x)$ is also in $\mathcal{N}^*$ (respectively $\mathcal{P}^*$).*

Example 4.1

A classical example of a query that can be constructed from a flat relational algebra query using the lfp operator, is the *transitive closure* of a binary flat relation. We show that this query can be expressed in the lfp closure of the nested algebra. Let $r = (\{A, B\}, \omega)$. Consider the following lfp expression:

$$E(x) = x \cup \pi_{\{A,B\}} \sigma_{C=D}(\rho_{C \leftarrow B}(x) \times \rho_{D \leftarrow A}(x))$$

Obviously the transitive closure of r equals $E^*(r)$.

As the transitive closure of a binary relation cannot even be computed in the nested relational algebra [18], it follows that the *lfp closure of the nested algebra* is strictly more expressive than the ordinary nested algebra. Actually, it has been shown in [10] (and independently also in [2]) that the lfp closure of the nested algebra is equivalent to the powerset algebra:

Theorem 4.1

The lfp closure of the nested relational algebra, the powerset algebra and the lfp closure of the powerset algebra have the same expressive power.

In the following two sections, we propose some other possible extensions of the nested relational algebra and show their equivalence to the powerset algebra. Because of space limitations, most of the proofs are either omitted or replaced by a sketch. Interested readers can find more details in [11].

5. Introducing while-loops in the algebra

In [4], Chandra and Harel introduce the language *QL* to express queries on flat relations. *QL* basically consists of the classical flat relational algebra, augmented

with two very powerful features: *unranked variables* (i.e. not associated with a fixed scheme) and a *while-construct*. Both features give to QL the computing power of Turing Machines and hence all computable queries on flat relations can be expressed in QL.

Of course, we should not hope that e.g. the powerset algebra would have the same expressive power as QL, since powerset algebra expressions have a fixed scheme. In [5] however, Chandra and Harel introduced and studied the language RQL which is syntactically almost identical to QL, but with the restriction that all variables are ranked (i.e. associated to a fixed scheme). Therefore, we wanted to see what happens to the expressive power of the nested algebra, when it is augmented with a while-construct. We shall show in this section that this augmentation results in exactly the expressiveness of the powerset algebra, provided slight extensions are made to deal with undefinedness.

5.1. The υ-operator

In Definition 2.2.2, we defined expressions in such a way that their result for some finite sequence of relations could never be undefined; "illegal" expressions or "illegal" substitutions were simply not allowed. The situation for while-loops however is clearly different; although not yet formally introduced, one can readily see that the result of a while-loop can be undefined in case it runs indefinitely. Moreover, the result of a while-loop being undefined can really depend on the actual instances used to start up the while-loop. Therefore, we shall extend the nested algebra (respectively the powerset algebra) slightly to deal with this situation.

First, for each composed attribute $X \in \mathcal{U} - U$, we add to $\mathcal{I}_X$, the set of all relation instances over X, the new value $?_X$, which should be interpreted as the "undefined instance" over the scheme X. Note that, by this convention, the values $?_X$ *cannot* appear on a lower level in a nested relation. Of course, we now must point out how the basic operators of the nested algebra (respectively the powerset algebra) work on this new values. Very straightforwardly, we agree that whenever a basic operator works on a relation (having the appropriate scheme) with an "undefined instance", then the result is also a relation with an "undefined instance", the scheme of which is defined by the operation.

Example 5.1.1

Let (Ω,ω) be a relation. Let $\Omega' \in \mathcal{U} - U$ have no elementary attributes in common with Ω. Assume furthermore that $X \subseteq \Omega$. Then:

- $(\Omega,?_\Omega) \cup (\Omega,\omega) = (\Omega,?_\Omega)$;
- $(\Omega,\omega) \times (\Omega',?_\Omega) = (\Omega \cup \Omega',?_{\Omega\cup\Omega'})$;
- $\nu_X(\Omega,?_\Omega) = ((\Omega - X) \cup \{X\},?_{(\Omega-X)\cup\{X\}})$. ∎

Obviously, we need an operator that can generate an "undefined instance" in an input-dependent way:

Definition 5.1.1

Let (Ω,ω) be a relation. The υ-operator is defined by:

$$\upsilon(\Omega,\omega) = \begin{cases}(\Omega,\omega) & \text{if } \omega \neq \emptyset \\ ?_\Omega & \text{if } \omega = \emptyset\end{cases}$$
$$\upsilon(\Omega,?_\Omega) = ?_\Omega$$

∎

In words, the υ-operator (where "υ" stands for "undefined") maps each non-empty relation to itself and the empty relation to that newly introduced value which corresponds to the scheme of the relation.

We now consider the *extended nested algebra* (respectively the *extended powerset algebra*) whose set of expressions is defined by the basic operators of the nested algebra (respectively the powerset algebra), augmented with the υ-operator. These generalized expressions are called *extended naes (enaes)* (respectively *extended paes (epaes)).*

Expressions in which the υ-operator occurs are in general rather difficult to handle. Fortunately, it will turn out that we only have to consider a subclass of all enaes (respectively epaes) to generate the extended nested algebra (respectively the extended powerset algebra). In preparation of this normalization, we first define the *global unnesting* and the *global nesting* operator.

Definition 5.1.2

- *Let $(\{\Omega\},\omega)$ be a relation. The global unnesting $M(\{\Omega\},\omega)$ equals $\mu_\Omega(\{\Omega\},\omega)$;*
- *Let (Ω,ω) be a relation. Then the global nesting $N(\Omega,\omega)$ equals:*

$$N(\Omega,\omega) = \begin{cases}(\{\Omega\},?_{\{\Omega\}}) & \text{if } \omega = ?_\Omega \\ (\{\Omega\},\{\omega\}) & \text{otherwise}\end{cases}$$

∎

Lemma 5.1.1

There exists naes (and hence enaes) that express the global unnesting and the global nesting. ∎

Since the global nest operator never returns an emtpy relation, even if the relation on which it is applied is empty, it can be used to "shield" relations from unwanted side-effects of the υ-operator. In this way, it is possible to pull back an occurrence of the υ-operator to the end of the expression it appears in. Finally, global unnesting can then be used to undo the global nesting. These ideas motivate the following definition:

Definition 5.1.3

A normalized enae (respectively epae) is an enae (respectively epae) $M\upsilon E(x,y,\ldots)$, where $E(x,y,\ldots)$ is a nae (respectively pae). ∎

Lemma 5.1.2

The set of all normalized enaes (respectively epaes) has the same expressive power as the extended nested algebra (respectively the powerset algebra).

Proof: By straightforward induction. ∎

5.2. While-loop queries

In this subsection, we shall introduce while-loops in the algebra. Therefore we first define in a formal way what we understand by a *while-loop query*:

Definition 5.2.1

Let $r_1 = (\Omega_1, \omega_1)$ and $r_2 = (\Omega_2, \omega_2)$ be relations. Let $E_1(y)$ be a unary expression, defined on relations with scheme Ω_2, and let $E_2(x,y)$ and $E_3(x,y)$ be binary expressions, defined on pairs of relations over the schemes Ω_1 and Ω_2 respectively, for which Ω_1 respectively Ω_2 are the schemes of the resulting relations. A while-loop query of type (Ω_1, Ω_2) is a binary query Q for which $Q(r_1, r_2)$ can be written as the result of a program of the following form:

begin
  $r := r_1$;
  $s := r_2$;
  while $E_1(s) \neq (\Omega^{E_1}, \emptyset)$
    do
      $r := E_2(r,s)$;
      $s := E_3(r,s)$
    od;
  $Q(r_1, r_2) := r$
end.

We assume that $Q(r_1, r_2)$ takes the value $(\Omega_1, ?_{\Omega_1})$ if the while-loop runs indefinitely. ∎

In the while-loop query above, the actual computation is done by the expression $E_2(x,y)$; $E_3(x,y)$ is used to compute the test relation whereas $E_1(y)$ expresses the test condition. In Definition 5.2.1, we chose a test of the form $E(s) \neq (\Omega^E, \emptyset)$. One might also have chosen: $E(s) \neq E'(s)$, $E(s) = E'(s)$, $E(s) = (\Omega^E, \emptyset)$, $E(s) = (\Omega^E, \{\emptyset\})$, $E(s) \neq (\Omega^E, \{\emptyset\})$, $s \neq \emptyset$, $s = \emptyset$, ... We leave it to the reader to convince himself that all these choices are equivalent with respect to expressive power.

Note that, since, apart from possibly "undefined instances", no new values are introduced during the execution of the while-loop above, it is decidable whether or not a while-loop will end.

If the expressions in Definition 5.2.1 are all enaes (respectively epaes), we say that Q is a *while-loop query in the extended nested algebra* (respectively *the extended powerset algebra*). By starting from enaes (respectively epaes) and by recursively associating a binary expression to each while-loop query, one can define the *while-closure of the extended nested algebra* (respectively *the extended powerset algebra*).

Obviously, the while-closure of the extended nested algebra is more powerful than the extended nested algebra itself:

Example 5.2.1

We show that the transitive closure of a binary flat relation can be expressed in the while-loop closure of the nested algebra. Let $E(x)$ be the expression defined in Example 4.1. Now consider the following while-loop query of type $(\{A,B\}, \{A,B\})$:

begin
  $r := r_1$;
  $s := r_2$;
  while $E(s) - s \neq (\{A,B\}, \emptyset)$
    do
      $r := E(r)$;
      $s := E(s)$
    od;
  $Q(r_1, r_2) := r$
end.

Obviously, the transitive closure of a relation $r_1 = (\{A,B\}, \omega)$ can be expressed as $Q(r_1, r_1)$. ∎

5.3. The expressiveness of the while-construct

In this subsection, we show that the while-closure of the extended nested relational algebra, the powerset algebra and the while-closure of the extended powerset algebra have the same expressive power.

Theorem 5.3.1

The while-closure of the extended nested algebra, the extended powerset algebra and the while-closure of the extended powerset algebra have the same expressive power.

Proof: We only give a sketch. The theorem will be proven if we can demonstrate:

1. The powerset operator can be expressed by a while-loop query in the extended nested algebra;
2. Each while-loop query in the extended powerset algebra can be expressed in the extended powerset algebra.

The first item is the easier to show: Let $r_1 = (\Omega_1, \omega_1)$ be an arbitrary relation. First, we mention that it is possible to construct an nae $E_1(x)$ such that $E_1(r_1)$ is a relation with scheme $\{\Omega_1\}$ that consists of all subsets of ω of size at most 1. Suppose now that we are given a relation $r = (\{\Omega_1\}, \mathcal{W})$. It is also possible to construct an nae $E_2(y)$ such that $E_2(r) = (\{\Omega\}, \{w_1 \cup w_2 \mid$

$w_1, w_2 \in W\}$. Hence, if r consists of all subsets of r_1 up to size i, then $E_2(r)$ consists of all subsets of r up to size $2i$. It is now straightforward to construct a while-loop query Q such that $\Pi(r_1) = Q(E_1(r_1), E_1(r_1))$.

The second part of the proof is much trickier. We only give the main ideas here. First of all, it is important that we may assume without loss of generality that all the expressions in a while-loop query are normalized. Finally, by subtilly using global nesting, the powerset operator and the least fixpoint operator, we can define an expression that constructs the set of all intermediate results of the while-loop query. Then a selection according to the condition of the while-loop will yield the correct result; if no relation in the aforementioned set satisfies the condition, an application of the v-operator will yield an "undefined" relation as output of the while-loop query. ∎

5.4. The if-then-else construct

We might also want to introduce an if-then-else construct in the extended nested algebra (respectively the extended powerset algebra):

Definition 5.4.1

Let $r_1 = (\Omega_1, \omega_1)$ and $r_2 = (\Omega_2, \omega_2)$ be relations. Let $E_1(y)$ be a unary expression, defined on relations with scheme Ω_2, and let $E_2(x)$ and $E_3(x)$ be unary expressions, defined on relations with scheme Ω_1 such that $\Omega^{E_2} = \Omega^{E_3}$. An if-then-else-query of type (Ω_1, Ω_2) is a binary query Q for which $Q(r_1, r_2)$ can be written as the result of a program of the following form:

begin
  if $E_1(r_2) \neq (\Omega^{E_1}, \emptyset)$
    then
      $Q(r_1, r_2) := E_2(r_1)$
    else
      $Q(r_1, r_2) := E_3(r_1)$
end.

∎

We have [9]:

Theorem 5.4.1

If-then-else queries in the extended nested algebra (respectively the extended powerset algebra) can be expressed in the extended nested algebra (respectively the extended powerset algebra). ∎

6. Introducing for-loops in the algebra

In the beginning of the previous section, we discussed the language QL introduced in [4]. Due to the fact that variables are not ranked, i.e that the scheme of the relation they represent can grow wider during the computation process, it is possible to simulate counting in QL, which, in combination with the presence of while-loops gives QL the power of general Turing Machines. Of course we cannot expect the same for the powerset algebra, where all expressions are ranked. However, we still retain some of that, since in this section we shall show that a certain type of for-loop can be expressed in the powerset algebra.

Definition 6.1

Let $r_1 = (\Omega_1, \omega_1)$ and $r_2 = (\Omega_2, \omega_2)$ be relations. Let $E_1(x)$ be a unary scheme preserving expression, defined on relations with scheme Ω_1. A for-loop query of type (Ω_1, Ω_2) is a binary query Q for which $Q(r_1, r_2)$ can be written as the result of a program of the following form:

begin
  $r := r_1$;
  for $i := 1$ **to** $|r_2|$
    do
      $r := E_1(r)$;
    od;
  $Q(r_1, r_2) := r$
end.

In this program $|r_2|$ stands for $|\omega_2|$, i.e. the number of tuples of r_2. We assume that $Q(r_1, r_2)$ takes the value $(\Omega_1, ?_{\Omega_1})$ if $\omega_2 = ?_{\Omega_2}$. ∎

Example 6.1

Again, let us have a look to the transitive closure of a binary flat relation. Let $E(x)$ be the expression defined in Example 4.1. Consider the following for-loop query of type $(\{A, B\}, \{A, B\})$:

begin
  $r := r_1$;
  for $i := 1$ **to** $|r_2|$
    do
      $r := E(r)$;
    od;
  $Q(r_1, r_2) := r$
end.

Now let $E'(x) = \pi_{\{A\}}(x) \times \pi_{\{B\}}(x)$. Obviously, the transitive closure of a relation $r_1 = (\{A, B\}, \omega)$ can be expressed as $Q(r_1, E'(r_1))$. ∎

As in the previous section, we can define the *for-closure of the extended nested algebra* (respectively *the extended powerset algebra*). We are now going to show that both closures are equivalent to the extended powerset algebra.

Theorem 6.1

The for-closure of the extended nested algebra, the extended powerset algebra and the for-closure of the extended powerset algebra have the same expressive power.

Proof: Again, the easiest direction is showing that the powerset operator can be expressed using a for-loop query in the extended nested relational algebra. Let $r_1 = (\Omega_1, \omega_1)$. Reconsider the proof of Theorem 5.3.1. Since the while-loop constructed in the first part of that proof needs to cycle at most $|r_1|$ times (this is a very rough estimate) to compute the desired result, it is straightforward to transform that while-loop into a for-loop.

Now let $r_1 = (\Omega_1, \omega_1)$ and $r_2 = (\Omega_2, \omega_2)$ be relations and reconsider the for-loop query Q of Definition 5.1 in which the expressions are supposed to be epaes. We show that this query can be expressed in the powerset algebra, or, equivalently, in the while-loop closure of the powerset algebra. First, let $r = (\{\Omega_2\}, \mathcal{W})$. There exists an enae $E_2(y)$ such that $E_2(r)$ is a relation with scheme $\{\Omega_2\}$ consisting of those elements of $\mathcal{W}$ that are *not* minimal with respect to inclusion. In particular, it follows that $|r_2|$ is the smallest integer i for which $E_2^i(\Pi(r_2) - (\{\Omega_2\}, \{\emptyset\})) = \emptyset$. It is now straightforward to construct a while-loop query Q' such that $Q'(r_1, r_2) = Q(r_1, \Pi(r_2) - (\{\Omega_2\}, \{\emptyset\}))$. ∎

Acknowledgment

The first author is a senior research assistant of the Belgian National Fund of Scientific Research. He also wishes to acknowledge the financial support of IBM Belgium, which enabled him to visit Indiana University, where part of this joint research was performed.

References

[1] S. Abiteboul, C. Beeri, "On the Power of Languages for the Manipulation of Complex Objects", *draft*, April 1987.

[2] S. Abiteboul, C. Beeri, *personal communications.*

[3] A.V. Aho, J.D. Ullman, "Universality of Data Retrieval Languages", *Proc. 6th POPL*, San-Antonio, 1979, pp. 110–117.

[4] A.K. Chandra, D. Harel, "Computable Queries for Relational Data Bases", *Journal of Computer and System Sciences 21*, 1980, pp. 156–178.

[5] A.K. Chandra, D. Harel, "Structure and Complexity of Relational Queries", *Journal of Systems and Computers Sciences 25*, 1985, pp 99–128.

[6] A.K Chandra, D. Harel, "Horn Clause Queries and Generalizations", *Journal of Logic Programming 1*, 1985, pp. 1–15.

[7] E.F. Codd, "A Relational Model of Data for Large Shared Data Banks", *Communications of ACM 13:6*, June 1970, pp. 377–387.

[8] M. Gyssens, "The extended nested relational algebra", *Techn. Rep. 87-11*, Univ. of Antwerp, 1987.

[9] M. Gyssens, G. Houben, J. Paredaens, D. Tahon, *unpublished results.*

[10] M. Gyssens, D. Van Gucht, "The Powerset Operator as an Algebraic Tool for Understanding Least Fixpoint Semantics in the Context of Nested Relations", *Techn. Rep. 233*, Indiana Univ., Bloomington, 1987.

[11] M. Gyssens, D. Van Gucht, "The Powerset Algebra as a Result of Adding Programming Constructs to the Nested Relational Algebra", *Techn. Rep. 87-26*, Univ. of Antwerp, 1987.

[12] G. Houben, J. Paredaens, D. Tahon, *personal communications.*

[13] R. Hull, "On the Expressive Power of Database Queries with Intermediate Types", *Techn. Rep.*, Univ. of South. California, Los Angeles, 1987.

[14] G. Jaeschke, H.J. Schek, "Remarks on the Algebra on Non First Normal Form Relations", *Proc. 1st PODS*, Los Angeles, 1982, pp. 124–138.

[15] G.M. Kuper, M.Y. Vardi, "A New Approach to Database Logic", *Proc. 3rd PODS*, Waterloo, 1984, pp. 86–96.

[16] A. Makinouchi, "A Consideration of Normal Form of Not-Necessarily-Normalized Relations in the Relational Data Model", *Proc. 3rd VLDB*, Tokyo, 1977, pp. 447–453.

[17] G. Özsoyoğlu, Z.M. Özsoyoğlu, V. Matos, "Extending Relational Algebra and Relational Calculus with Set-Valued Attributes and Aggregate Functions", *ACM Transactions On Database Systems 12:4*, Dec. 1987, pp. 566–592.

[18] J. Paredaens, D. Van Gucht, "Possibilities and Limitations of Using Flat Operators in Nested Algebra Expressions", *Proc. 7th PODS*, Austin, 1988.

[19] M.A. Roth, H.F. Korth, A. Silberschatz, "Theory of Non-First-Normal-Form Relational Databases", *Techn. Rep. TR-84-36*, Univ. of Texas, Austin, 1986.

[20] A. Tarski, "A Lattice-Theoretical Fixpoint Theorem and its Applications", *Pacific Journ. of Mathematics 5*, 1955, pp. 285–309.

[21] S.J. Thomas, P.C. Fischer, "Nested Relational Structures", *The Theory of Databases*, P.C. Kanellakis, ed., JAI Press, 1986, pp. 269–307.

[22] D. Van Gucht, "On the Expressive Power of the Extended Relational Algebra for the Unnormalized Relational Model", *Proc. 6th PODS*, San Diego, 1987, pp. 302–312.