

A COMPLETE IDENTITY SET FOR CODD ALGEBRAS
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Introduction

In his book (1) on the relational model version2, Codd defines a 4-valued propositional logic with two different valuations for missing information, missing-and-applicable, and missing-and-inapplicable. The truth tables are the following (+ is disjunction, * conjunction, N() negation):

x	N(x)	x+y	t a i f	x*y	t a i f
t	f	t	t t t t	t	t a i f
a	a	a	t a a a	a	a a i f
i	i	i	t a i f	i	i i i f
f	t	f	t a f f	f	f f f f

The algebraization of classical 2-valued propositional logic delivers Boolean Algebras, which are complemented distributive lattices. The algebraization of 3-valued propositional logic gives still distributive lattices (but not complemented), whereas the corresponding algebras in the case of Codd's 4-valued propositional logic are not even lattices.

The purpose of this article is to give a representation theorem for these Codd Algebras, in analogy to the representation theorem for Boolean Algebras, which says that every Boolean Algebra is a subalgebra of a product of 2-element algebras.

It is assumed that the reader has some elementary knowledge in universal algebra, which can be obtained from any textbook on the subject, for instance (2).

Definitions

We consider the language $L = \{t, a, i, f, N, +, *\}$, where the (nonlogical) symbols t, a, i, f are constants, and N is a unary operation symbol and $+$ and $*$ are binary operation symbols.

Let the Simple Codd Algebra be (up to isomorphism) the algebra of the language L with four elements, which are the interpretations of the constants t, a, i, f , and with the operations $N, +, *$ interpreted according to the abovementioned operation tables.

A Codd Algebra is an algebra for the language L in which the same identities of L hold true that hold true in the Simple Codd Algebra.

Let V be the following set of identities of the language L :

$t+x=t$	$t*x=x$	Id1		
$i+x=x$	$f*x=f$	Id2		
$a+f=a$	$a*i=i$	Id3		
$x+y=y+x$	$x*y=y*x$	Id4		
$x+(y+z)=(x+y)+z$	$x*(y*z)=(x*y)*z$	Id5		
$x+x=x$	$x*x=x$	Id6		
$x+(y*z)=(x+y)*(x+z)$		Id7		
$N(t)=f$	$N(a)=a$	$N(i)=i$	$N(f)=t$	Id8
$N(N(x))=x$				Id9
$N(x)+z=N(x+z)+z$	where $z=N(i*y)$			Id10
$N(x*i)*i=i$				Id11
$N((N((z+f)*i)+f)*i)=z$	where $z=N(i*x)$			Id12
$x=(N(N(x)*z)*z)+(N(N(x)*w)*w)$	where $z=N(i*y)$ and $w=N((z+f)*i)$			Id13
$N(N(x)*z)*z=N(N(x+w)*z)*z$	where $z=N(i*y)$ and $w=N((z+f)*i)$			Id14
$x*i=N(x*i)*x$				Id15
$x+f=((x+f)*i)+x$				Id16
$(N((x+f)*i)+x)*a=a$				Id17
$N(x+f)+f=N(x+f)$				Id18
$x=(x*a)+x$				Id19
$N(N(x+f)+a)=(x+f)*a$				Id20
$N((x+f)*i)*x=x*i$				Id21

Observe that all identities in V hold true in the Simple Codd Algebra (and therefore, by definition, in all Codd Algebras). Note also that De Morgan's laws, the other distributive law, and the absorbtive laws do not hold true in Codd Algebras.

The representation theorem

Lemma1.

If A is a Codd Algebra and b an element from A with $b*i=i$, $N(b)*i=N(b)$, $b<>t$ (unequal) and $b<>i$, and if the relation q is defined by (x,y) in q iff $x+b=y+b$, for x,y in A , then q is a congruence relation.

Proof:

q clearly is an equivalence relation. Let (x,y) in q and z in A . From Id7,4,5 follows that $(x*z,y*z)$ in q and $(x+z,y+z)$ in q . Id9 gives $b=N(i*N(b))$, therefore with Id10 $N(x)+b=N(x+b)+b$, therefore $(N(x),N(y))$ in q .

Lemma2.

If A is a Codd Algebra and b an element from A with $b*i=i$, $N(b)*i=N(b)$, $b<>t$ and $b<>i$, then there are two nontrivial separating congruences q and r on A .

Proof:

Let $c=N((b+f)*i)$. With Id9 we have $b=N(i*N(b))$, therefore with Id11 $N((b+f)*i)*i=i$, which means $c*i=i$. With Id9,5,6 we get $N(c)*i=N(c)$. Id12 gives $N((N((b+f)*i)+f)*i)=b$, which means $b=N((c+f)*i)$, therefore $c<>t$ and $c<>i$.

Now let (x,y) in q iff $x+b=y+b$,
 and (x,y) in r iff $x+c=y+c$.
 By Lemma1, q and r are congruences on A .
 Id2,6 give (i,b) in q and (i,c) in r . Id1,6
 and $b < t$, $c < t$ show that (t,b) is not in q
 and (t,c) is not in r . So q and r are non-
 trivial.
 It remains to show that q and r are
 separating,
 e.g. if (x,y) in q and in r , then $x=y$.
 Now with $b=N(i*N(b))$, $c=N((b+f)*i)$, and
 Id13,14 we have

$$x=(N(N(x)*b)*b)+(N(N(x)*c)*c), \quad (I)$$

$$N(N(x)*b)*b=N(N(x+c)*b)*b, \quad (II)$$
 and, by symmetry,

$$N(N(x)*c)*c=N(N(x+b)*c)*c. \quad (III)$$
 So let (x,y) be in q and in r .
 This means $x+b=y+b$ and $x+c=y+c$. From (I),
 (II) and (III) it follows that $x=y$.

Corollary to Lemma2.
 If A is a Codd Algebra and b an element from
 A with
 $b*i=i$, $N(b)*i=N(b)$, $b < t$ and $b < i$, then A is
 subdirectly reducible.

Lemma3.
 Let A be a subdirectly irreducible Codd Al-
 gebra. Then for all x in A
 if $x < f$ then $x*i=i$, and (I)
 if $x+f < f$ then $x+f=x$. (II)

Proof:
 For (I) let A be any Codd Algebra and x in
 A , $x < f$, and $x*i < i$.
 Let $b=N(x*i)$.
 Then $b*i=i$ by Id11, and $N(b)*i=N(b)$ by
 Id9,5,6. By $x*i < i$ and Id8 we have $b < i$. We
 also have $b < t$ because otherwise, by
 Id9,15,8,1, $f=x*i=N(x*i)*x=t*x=x$.
 Now (I) follows from the Coroll. to Lemma2.
 For (II) let A be any subdirectly irreducib-
 le Codd Algebra and let x in A with $x+f < f$.
 Then, by (I), $(x+f)*i=i$, therefore, by
 Id16,2,4, $x+f=f$.

Lemma4.
 Let A be a subdirectly irreducible Codd Al-
 gebra, and x in A with $x=x+f$.
 Then $x=t$ or $x=a$ or $x=f$.

Proof:
 Let the Codd Algebra A be subdirectly irre-
 reducible, and x in A with $x < t$, $x < f$, and
 $x=x+f$.
 Then, by Lemma3, $x*i=i$ and $N(x)*i=i$. There-
 fore, with Id17,8,2 we have $a=x*a$. By Id18,
 $N(x)=N(x)+f$, therefore, analogously,
 $a=N(x)*a$. Therefore, by Id19, $N(x)=a+N(x)$,
 and, by Id9,20, $x=N(N(x)+a)=N(N(x+f)+a)=$
 $= (x+f)*a=x*a=a$.

Lemma5.
 Let A be a subdirectly irreducible Codd Al-
 gebra, and x in A with $x+f=f$.
 Then $x=i$ or $x=f$.

Proof:
 Let the Codd Algebra A be subdirectly irre-
 reducible, and x in A with $x < f$ and $x=x+f$.
 Then, by Lemma3, $x*i=i$, therefore, by
 Id21,2,8,1, $i=x*i=N((x+f)*i)*x=N(f*i)*x=$
 $=N(f)*x=t*x=x$.

Corollary to Lemma3,4,5.
 Let A be a subdirectly irreducible Codd Al-
 gebra, and x in A any element.
 Then $x=t$ or $x=a$ or $x=i$ or $x=f$.

In view of the Birkhoff theorem on subdirect
 reducibility, this Corollary amounts to the
 following representation theorem.

Representation theorem.
 Every Codd Algebra is isomorphic to a sub-
 direct product of copies of the simple Codd
 Algebra.

Corollary to the representation theorem.
 The identity set V is complete for Codd Al-
 gebras, e.g. any identity of the language L
 which holds true in all Codd Algebras, can
 be derived from the identities of V .

Proof:
 This is a standard argument. One has to ob-
 serve that in the above construction only
 identities of V are used, and that in a sub-
 direct product of copies of an algebra the
 same identities are true as in this algebra.

Concluding remarks

- 1) It is an almost trivial exercise to write
 down a set Q of Horn formulas such that
 the variety of the Codd Algebras defined
 by V is also the quasivariety defined by
 Q .
 Hint: Consider the definable function
 $K(x)=N((x+f)*i)$. Then A is irreducible
 iff for all x in A , $K(x)=t$ or $K(K(x))=t$.
- 2) With the results of (3) it is also an
 easy exercise to show that the elementary
 first order theory of Codd Algebras is
 decidable.
 Hint: Whereas MV (the algebraization of
 many-valued Lukasiewicz propositional
 logic) is undecidable, finite order MV
 (in (3) $S2, S3, S4, \dots$) is decidable. Find a
 faithful interpretation of V in an ex-
 tension by definitions of $S4$ (use the
 representation theorems).
- 3) In view of possible structures of opti-
 mizers for relational query languages it
 would be interesting to have a good theo-
 ry of normal forms for terms in Codd Al-
 gebras (but it might be possible that
 there are no "esthetic" normal forms).

Literature

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