Some Further Analysis of the Essential Blocking Recurrence

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In previous work,[1] a random graph model of concurrency control was developed, in which there are n concurrent transactions (represented as vertices of a graph), and where for each pair of transactions (vertices), a conflict (edge between the vertices) initially occurs independently with probability p. Given such a graph, a *scheduling function* selects a subset of transactions to complete based on the conflicts and possibly on an ordering of the transactions. At the end of each unit of time, the transactions selected by the scheduling function to complete are removed from the graph, and are replaced by new transactions from a transaction sequence. For each pair of transactions in the new graph, if the transactions were both in the previous graph then a conflict occurs between them if and only if there was a conflict in the previous graph; otherwise one or both of the transactions are new and a conflict occurs independently with probability p.

In order to investigate the limits of transaction processing for arbitrarily increasing degrees of concurrency n, various scheduling functions were studied analytically, where the scheduling functions were defined so as to capture key properties of various classes of actual concurrency control methods (the analytic results were confirmed by means of simulations of a more detailed model of transaction processing systems). One such class of concurrency control methods was called *essential blocking with priority*, defined as follows: (1) a transaction is doing *useful work* if it is running and will not be subsequently aborted; and (2) a transaction is blocked from doing useful work (by waiting or by being aborted) if and only if it conflicts with some higher priority transaction that is doing useful work. For example, dynamic locking is not in this class (since one transaction can be waiting on another waiting transaction, i.e. waiting on a transaction not doing useful work), but optimistic methods and static locking are (however, static locking is not generally useful since all lock requests must be known in advance). The scheduling function for this class was defined recursively as follows: numbering the transactions T_i according to priority (with T_1 having highest priority), (1) T_1 completes; and (2) T_i completes if and only if there is no $j < i$ such that T_j has been selected to complete and T_j conflicts with T_i. This class proved to be of interest since of the various classes considered, it was the only class of practical methods in which under the random graph model the expected number of completing transactions was not bounded for increasing n.

Using the above scheduling function, the following recurrence was derived for the probability a_i that transaction T_i completes:

$$a_1 = 1; \quad a_{i+1} = a_i(1 - pa_i).$$

Here this is called the *essential blocking recurrence*. It was shown that

$$\frac{1-p}{p(i-2)+1} \le a_i \le \frac{1}{p(i-1)+1}.$$

In this note we show the derivation of a more accurate approximation for a_i. The result is primarily of mathematical interest since it is easy to compute a_i to any desired degree of accuracy (for practical values of i) using the above recurrence, and furthermore the above bounds are sufficient to show that the sum of the a_i for $1 \le i \le n$ (the quantity of interest in practice) grows as $\log(n)$.[1]

First, let $A_i = 1/a_i$; we have:

$$A_1 = 1;$$

$$A_{i+1} = A_i \frac{1}{1 - p/A_i}$$

$$= A_i + \frac{p}{1 - p/A_i};$$

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[1] Franaszek, P. and Robinson, J. T. Limitations of Concurrency in Transaction Processing, *ACM Trans. Database Systems 10*, 1 (March 1985), 1-28.

$$A_{i+1} - A_i = p\left(1 + \frac{p}{A_i} + \left(\frac{p}{A_i}\right)^2 + \dots\right).$$

From this it can be seen that A_i is of the form

$$A_i = 1 + p(i-1) + B_i,$$

where $B_1 = 0$, $B_i > 0$ for $i > 1$ (note that the upper bound for a_i previously derived by different methods follows immediately from this). Numerical experiments show that setting B_i to 0 typically gives a reasonable approximation for A_i (with increasing accuracy for decreasing p); call this approximation $A_i^{(1)}$:

$$A_i \cong A_i^{(1)} = 1 + p(i-1).$$

In order to get a more accurate approximation we need to estimate B_i. A recurrence for B_i can be derived as follows:

$$\begin{aligned}
B_{i+1} - B_i &= (A_{i+1} - 1 - p\,i) - (A_i - 1 - p(i-1)) \\
&= A_{i+1} - A_i - p \\
&= \frac{p}{1 - p/A_i} - p \\
&= \frac{p^2/A_i}{1 - p/A_i} \\
&= \frac{p^2}{A_i - p} \\
&= \frac{p^2}{1 + p(i-2) + B_i}.
\end{aligned}$$

This shows that the B_i grow approximately like the harmonic numbers; the problem is the B_i in the denominator of the right hand side of this recurrence. However, for small p, B_i is small compared to the remaining $1 + p(i-2)$ term in the denominator (for small i and p, $B_i \cong (i-1)p^2$, and for increasing i, B_i grows at a lesser rate). Therefore in order to estimate B_i we will set B_i to 0 in the right hand side of the recurrence:

$$B_{i+1} - B_i \cong \frac{p^2}{1 + p(i-2)} = p\left(\frac{1}{i-2+1/p}\right).$$

Next let $P = 1/p$ and assume that P is an integer; this gives:

$$\begin{aligned}
B_{i+1} - B_i &\cong p\left(\frac{1}{i-2+P}\right) \\
&= p\,(H_{P+i-2} - H_{P+i-3}).
\end{aligned}$$

From this it follows that

$$\begin{aligned}
B_i &\cong p\,(H_{P+i-3} - H_{P-2}) \\
&\cong p\left(\ln\left(\frac{P+i-3}{P-2}\right) + \frac{1}{2(P+i-3)} - \frac{1}{2(P-2)}\right) \\
&= p\ln\left(\frac{1+p(i-3)}{1-2p}\right) + \frac{p^2}{2+2p(i-3)} - \frac{p^2}{2-4p},
\end{aligned}$$

where the approximation $H_k \cong \gamma + \ln(k) + 1/2k$ has been used. This gives an improved approximation $A_i^{(2)}$ for A_i:

$$A_i \cong A_i^{(2)} = 1 + p(i-1) + p\ln\left(\frac{1+p(i-3)}{1-2p}\right) + \frac{p^2}{2+2p(i-3)} - \frac{p^2}{2-4p}.$$

Define $a_i^{(1)} = 1/A_i^{(1)}$ and $a_i^{(2)} = 1/A_i^{(2)}$ as the corresponding approximations for a_i. Tables showing a_i, $a_i^{(1)}$, $a_i^{(2)}$, and the approximation errors are given below for $p = 0.1, 0.01$ and for $i = 1, 2, 3, \dots, 10, 20, 30, \dots, 100$. From the tables we see that $a_i^{(2)}$ is quite accurate, correct to three rounded decimal places or more even for the relatively high pairwise probability of conflict $p = 0.1$, and with five decimal place or more accuracy for $p = 0.01$, $i \le 100$.

Note that it is not necessary for P to be an integer; the expression $H_{P+i-3} - H_{P-2}$ actually represents the sum:

$$\frac{p}{1+(i-3)p} + \frac{p}{1+(i-4)p} + \dots + \frac{p}{1+2p} + \frac{p}{1+p} + p + \frac{p}{1-p}.$$

The approximation errors for $a_2^{(2)}$ in the tables (which would have been exact if we had used $B_2 = p(H_{P-1} - H_{P-2})$), and also the fact that $A_i^{(2)}$ is not defined for $p \ge 1/2$, result from using the logarithmic approximation for H_k above.

TABLE 1: p=0.1

i	a_i	$a_i^{(1)}$	$a_i^{(1)}-a_i$	$a_i^{(2)}$	$a_i-a_i^{(2)}$
1	1	1	0	1	0
2	0.900000	0.909090	0.009090	0.900022	-0.000022
3	0.819000	0.833333	0.014333	0.818957	0.000042
4	0.751923	0.769230	0.017306	0.751800	0.000123
5	0.695384	0.714285	0.018900	0.695186	0.000198
6	0.647028	0.666666	0.019637	0.646769	0.000259
7	0.605164	0.625000	0.019835	0.604857	0.000307
8	0.568541	0.588235	0.019693	0.568199	0.000342
9	0.536217	0.555555	0.019337	0.535851	0.000366
10	0.507464	0.526315	0.018850	0.507081	0.000383
20	0.331784	0.344827	0.013043	0.331428	0.000355
30	0.247289	0.256410	0.009120	0.247020	0.000269
40	0.197369	0.204081	0.006712	0.197165	0.000203
50	0.164335	0.169491	0.005156	0.164176	0.000158
60	0.140831	0.144927	0.004096	0.140705	0.000125
70	0.123242	0.126582	0.003340	0.123140	0.000102
80	0.109577	0.112359	0.002781	0.109493	0.000084
90	0.098653	0.101010	0.002356	0.098582	0.000070
100	0.089718	0.091743	0.002025	0.089657	0.000060

TABLE 2: p=0.01

i	a_i	$a_i^{(1)}$	$a_i^{(1)}-a_i$	$a_i^{(2)}$	$a_i-a_i^{(2)}$
1	1	1	0	1	0
2	0.9900000	0.9900990	0.0000990	0.9900000	-1.7093E-9
3	0.9801990	0.9803921	0.0001931	0.9801989	6.4026E-9
4	0.9705910	0.9708737	0.0002826	0.9705910	2.3289E-8
5	0.9611706	0.9615384	0.0003678	0.9611705	4.7997E-8
6	0.9519321	0.9523809	0.0004488	0.9519320	7.9659E-8
7	0.9428703	0.9433962	0.0005258	0.9428702	1.1748E-7
8	0.9339803	0.9345794	0.0005990	0.9339801	1.6075E-7
9	0.9252571	0.9259259	0.0006687	0.9252569	2.0880E-7
10	0.9166961	0.9174311	0.0007350	0.9166958	2.6105E-7
20	0.8390933	0.8403361	0.0012427	0.8390924	9.1590E-7
30	0.7736477	0.7751937	0.0015459	0.7736461	0.0000016
40	0.7177044	0.7194244	0.0017200	0.7177021	0.0000022
50	0.6693299	0.6711409	0.0018110	0.6693271	0.0000027
60	0.6270828	0.6289308	0.0018479	0.6270797	0.0000031
70	0.5898663	0.5917159	0.0018496	0.5898629	0.0000034
80	0.5568309	0.5586592	0.0018283	0.5568272	0.0000036
90	0.5273083	0.5291005	0.0017921	0.5273045	0.0000037
100	0.5007658	0.5025125	0.0017467	0.5007619	0.0000038